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WEIGHTED WEAK (1,1) AND WEIGHTED L^p ESTIMATES FOR OSCILLATING KERNELS

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ABSTRACT. Weak type $(1,1)$ and strong type (p,p) inequalities are proved for operators defined by oscillating kernels. The techniques are sufficiently general to derive versions of these inequalities using weighted norms.

0. Introduction. Given a positive real number $a > 0$, $a \neq 1$, define the oscillating kernel K_a by

$$K_a(x) = (1+|x|)^{-1}e^{i|x|^a}$$

and consider the convolution operator $K_a * f$. In an earlier paper [**2**], we studied the boundedness properties of such operators on weighted L^p spaces, $1 < p < \infty$. For $p \geq 1$, we define

$$\|f\|_{p,w} = \left(\int_{\mathbf{R}} |f(x)|^p w(x)\,dx\right)^{1/p}$$

and say $f \in L^p_w(\mathbf{R})$ if $\|f\|_{p,w} < \infty$. The main result of this paper deals with an extension of these results to a weighted weak $(1,1)$ result. Additionally, we consider extensions of our results in [**2**]. Many of the ideas used here were developed in [**1**].

One standard technique for the study of convolution operators is to prove results for the corresponding multiplier. Suppose $0 < a \neq 1$ and $\beta = a/(a-1)$, so that $\beta < 0$ or $\beta > 1$. The multiplier operator T_β associated to K_a is defined by

$$(T_\beta f)\hat{}(\xi) = \theta(\xi)|\xi|^{-\beta/2}e^{i|\xi|^\beta}\hat{f}(\xi),$$

where θ is a C^∞ function defined by

(i) $$\theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1/2 \\ 0 & \text{if } |\xi| \geq 1 \end{cases} \quad \text{when } \beta < 0,$$

and

(ii) $$\theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \geq 1 \\ 0 & \text{if } |\xi| \leq 1/2 \end{cases} \quad \text{when } \beta > 1.$$

We will also need to consider the multiplier operator

$$(T_{\beta,c}f)\hat{}(\xi) = \theta(\xi)|\xi|^c e^{i|\xi|^\beta}\hat{f}(\xi),$$

where $\theta(\xi)$ is defined above. We shall denote $T_{\beta,-\beta/2}f$ by $T_\beta f$.

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Given a nonnegative, locally integrable weight function w, we say w satisfies the A_1 condition if there is a constant $C > 0$ such that for all intervals I

$$\frac{1}{|I|}w(I) = \frac{1}{|I|}\int_I w(x)\,dx \le C \operatorname*{ess\,inf}_I w$$

with C independent of I.

The main result of this paper is

THEOREM 1. *Let $\beta < 0$ or $\beta > 1$. If $w \in A_1$, there is a constant C, independent of f, so that for $\lambda > 0$*

$$w(\{x\colon |T_\beta f(x)| > \lambda\}) \le \frac{C}{\lambda}\|f\|_{1,w}. \tag{1}$$

An immediate consequence of this result is the following corollary.

COROLLARY 2. *Let $0 < a \ne 1$. If $w \in A_1$, there is a constant C, independent of f, so that for $\lambda > 0$*

$$w(\{x\colon |K_a * f(x)| > \lambda\}) \le \frac{C}{\lambda}\|f\|_{1,w}. \tag{2}$$

The two results (1) and (2) are actually equivalent.

We note for definiteness the case $w \equiv 1$ of the previous corollary.

COROLLARY 3. *Let $0 < a \ne 1$. There is a constant C, independent of f, so that for $\lambda > 0$*

$$|\{x\colon |K_a * f(x)| > \lambda\}| \le \frac{C}{\lambda}\|f\|_1.$$

Corollary 3 was proved for $0 < a < 1$ in [**8**]. The results when $a > 1$ are new.

Let $x \in \mathbf{R}^n$ and $k(x)$ be a Calderón-Zygmund kernel. Consider the operator Tf defined by

$$Tf(x) = \int_{\mathbf{R}^n} e^{iG(x,y)}k(x-y)f(y)\,dy.$$

Recently Phong and Stein [**9**] and Ricci and Stein [**10**] have considered the operator T as a mapping from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, $1 < p < \infty$, for certain types of functions G. Although we prove our results for $\mathbf{R}$ and $k(x) = 1/(1+|x|)$, an examination of the proofs shows the dimension may be greater than 1 and $1/(1+|x|)$ may be replaced by a Calderón-Zygmund kernel. The idea is to write

$$e^{i|x|^a}k(x) = \theta(x)e^{i|x|^a}k(x) + [1-\theta(x)]e^{i|x|^a}k(x) = K_1(x) + K_2(x),$$

where $\theta(x) = 1$ for $|x| \le 1$ and $\theta(x) = 0$ for $|x| \ge 2$. Next, write $K_1(x) = (e^{i|x|^a} - 1)\theta(x)k(x) + \theta(x)k(x)$. The first term is bounded by a constant times the L^1 function $\chi(\{|x| \le 2\})|x|^{a-n}$ and the second term is a Calderón-Zygmund kernel. Since $k(x)$ is homogeneous of degree $-n$, K_2 is handled like $e^{i|x|^a}(1+|x|)^{-n}$.

Ricci and Stein prove T defines a bounded operator on L^p when $G(x,y) = P(x,y)$ is a real-valued polynomial. If $a = 2k$ is an even integer, then $|x-y|^a = P(x,y)$ is a polynomial of degree $2k$ in x and y. Thus, Corollary 3 (and Corollary 2) gives weak type $(1,1)$ estimates for special classes of polynomials.

Suppose A is a nonsingular matrix and $G(x,y) = \langle Ax, y\rangle$. Phong and Stein show that T defines a bounded operator on L^p. When $a = 2$, $|x|^2 = \langle x, x\rangle$, so K_2 corresponds to the matrix $A = I$, the identity matrix. Our techniques can be used to prove the weak type $(1,1)$ inequality for a convolution operator with the kernel $K(x) = e^{i\langle Ax,x\rangle}k(x)$, where A is a nonsingular matrix and k is a Calderón-Zygmund kernel. The arguments are the same as those mentioned above with $e^{i|x|^a}$ replaced by $e^{i\langle Ax,x\rangle}$. Let A be symmetric and set $g(y) = e^{i\langle Ay,y\rangle}f(y)$. Since $\langle Ay,x\rangle + \langle Ax,y\rangle = 2\langle Ax,y\rangle$ and $|e^{i\langle Ax,x\rangle}| = 1$, we see that

$$\left|\int e^{i\langle A(x-y),x-y\rangle}k(x-y)f(y)\,dy\right| = \left|\int e^{2i\langle Ax,y\rangle}k(x-y)g(y)\,dy\right|.$$

Thus, for symmetric matrices A, the operators of Phong and Stein are weak type $(1,1)$.

To prove weak $(1,1)$ estimates for a Calderón-Zygmund kernel, one uses the fact that the derivative of the kernel belongs to L^1 away from the origin. We consider kernels $K_a(x)$ which have derivatives that decay like $|x|^{a-2}$ as $|x| \to \infty$. Thus, for $a > 1$, these kernels are not integrable away from the origin. Standard arguments which work for Calderón-Zygmund kernels cannot be used since they will fail for large intervals. The circle of ideas centered around the sharp function of C. Fefferman and E. M. Stein similarly do not apply. In fact, (at least) for $a > 1$, the kernels K_a are more akin to the Bochner-Riesz multipliers. We base our proof on the argument used to prove Theorem 3 in [**5**]. In order to introduce A_1 weights, we use a decomposition result from [**1**]. These ideas can be used to prove that operators defined by the kernels $(e^{it(\log t)^\eta}/t(\log t)^\delta)\chi(\{t \geq 2\})$ satisfy a weak type inequality when $\delta, \eta > 0$. These kernels were introduced in [**11**].

We will consider also strong type inequalities for operators with convolution kernels

$$K_{a,b}(x) = (1+|x|)^{-b}e^{i|x|^a}, \qquad 0 < a \neq 1 \text{ and } 1 - a/2 \leq b \leq 1.$$

Generalizing Theorem 3.3 of [**2**], we prove

THEOREM 4. *Let $a > 1$ and suppose $1 \leq w(x) \leq Cw(t)$ for $|t|/2 \leq |x| \leq 2|t|$. If either $1 - a/2 \leq b < 1$ and $a/(a+b-1) \leq p \leq 2$ or $b = 1$ and $1 < p \leq 2$, then*

$$\|K_{a,b}f\|_{p,w} \leq C\|f\|_{p,w}$$

if and only if

$$w(x) \leq C(1+|x|)^{a(p-1)+p(b-1)} \quad \text{for all } x.$$

This result is sharp for weights that are essentially constant on annuli. Note that for powers of $(1+|x|)$ the endpoint exponent $a(p-1)+p(b-1)$ is included.

This paper is divided into three sections. In §1 we give background information. The proof of Theorem 1 is contained in §2. We prove strong type results for operators with kernels $K_{a,b}$ in the last section.

We use the letters B and C to denote constants. Given an interval I, cI stands for the interval concentric with I and having length c times as long. Given a number p, $1 \leq p \leq \infty$, we define q by the equation $1/p + 1/q = 1$.

1. Preliminary estimates.

The relationship between $T_\beta f$ and $K_a f$, where $0 < a \neq 1$ and $\beta = a/(a-1)$, is expressed in the following lemma.

LEMMA 1.1. *Let $0 < a \neq 1$ and $\beta = a/(a-1)$. Then*

$$|T_\beta f(x) - K_a * f(x)| \leq \psi_a * |f|(x), \tag{1.1}$$

where $0 \leq \psi_a(x) \leq C(1+|x|)^{-1-a/2}$.

The proof for the case $a > 1$ is in [**12**] while that for $0 < a < 1$ follows from [**4**, p. 31(ii)]. In view of this lemma, $T_\beta f$ and $K_a f$ differ pointwise by the Hardy-Littlewood maximal function of f. Since the maximal function is weak $(1,1)$ with respect to A_1 weights, we may deduce Corollary 2 once Theorem 1 is proved.

The key estimate of this section is

LEMMA 1.2. *Let $w \in A_1$ and suppose $\beta < 0$ or $\beta > 1$. There is a δ, $0 < \delta < 1$, so that for each p, $2 - \delta < p < 2$, and $c = (p-2)\beta/2p$,*

$$\|T_{\beta,c} f\|_{p,w} \le C_p \|f\|_{p,w}.$$

We begin by considering the tools needed for the proof of this lemma. Much of the work in this direction involves estimates on operators mapping H^1_w into L^1_w.

The weighted Hardy spaces H^1_w, $w \in A_1$, have been developed in [**14**, **15**, **16**]. A real-valued function b is called an H^1_w atom if

(i) $b(x)$ is supported in an interval I,

(ii) $\int b(x)\,dx = 0$,

(iii) $\|b\|_\infty \le 1/w(I)$.

We say a function f is in H^1_w if there exist atoms $\{b_j\}$ and coefficients $\{\lambda_j\}$ such that $f(x) \sim \sum_j \lambda_j b_j(x)$ and $\sum_j |\lambda_j| < \infty$. Set $\|f\|_{H^1_w} = \inf \sum_j |\lambda_j|$, where the infimum is taken over all such decompositions.

The concept of regular kernels was considered in [**2**, **7**]. The following definition generalizes the idea to weights.

DEFINITION 1.3. A kernel K is regular with respect to a weight w if K can be decomposed as $K(x) = k(x)g(x)$ so that

$$|g(x)| \le B|g(u)| \quad \text{for } |u|/2 \le |x| \le 2|u|, \tag{1.2}$$

$$\|K * f\|_{2,w} \le B\|f\|_{2,w}, \tag{1.3}$$

and

$$\begin{aligned}\int_{\{|x|\ge 2|y|\}} &|k(x-y) - k(x)|\,|g(x)| w(x+h)\,dx \\ &\le \frac{B}{|y|}\int_{\{|x-h|\le |y|\}} w(x)\,dx, \quad \text{for all } y, h \in \mathbf{R}, y \ne 0.\end{aligned} \tag{1.4}$$

We note that the oscillating kernels K_a, $0 < a \ne 1$, are all regular with respect to A_1 weights. (See, e.g., [**7**] and Theorem 2.3 of [**2**].) The kernels $(1+|x|)^{-1-a/2}$, $a > 0$, are also regular with respect to A_1 weights. This follows by taking $g = 1$ in the definition above.

The next result is a generalization of Theorem 1 of [**7**].

THEOREM 1.4. *Let $K = kg$ be a regular kernel with respect to an A_1 weight w. Then*

$$\|K * f\|_{1,w} \le C\|f\|_{H^1_w}$$

if and only if for any H^1_w atom b with support $I = [d-\varepsilon, d+\varepsilon]$,

$$\int_{\{|x-d|\ge 2\varepsilon\}} |k(x-d)|\,|g * b(x)| w(x)\,dx \le B'.$$

Furthermore, the constant C depends only on B and B'.

PROOF. We first note that

$$K * b(x) = \int (k(x-t) - k(x-d))g(x-t)b(t)\,dt + k(x-d)g * b(x).$$

Thus, in order to show the equivalence of the two inequalities, it is enough to show

$$\int_{\{|x-d|<2\varepsilon\}} |K * b(x)| w(x)\,dx \le B \tag{1.5}$$

and

$$\int_{\{|x-d|\ge 2\varepsilon\}} \left| \int k(x-t)g(x-t)b(t)\,dt - k(x-d)g * b(x) \right| w(x)\,dx \le B. \tag{1.6}$$

By Hölder's inequality, (1.3), the definition of A_1 weights, and estimate (iii) in the definition of atoms, we have

$$\begin{aligned}\int_{\{|x-d|<2\varepsilon\}} |K * b(x)| w(x)\,dx &\le \left(\int_{\{|x-d|<2\varepsilon\}} w(x)\,dx \right)^{1/2} \|K * b\|_{2,w} \\ &\le B \left(\operatorname*{ess\,inf}_I w \right)^{1/2} |2I|^{1/2} \|b\|_{2,w} \\ &\le B \left(\operatorname*{ess\,inf}_I w \right)^{1/2} w(I)^{-1/2} |2I|^{1/2} \le \sqrt{2} B.\end{aligned}$$

This proves (1.5).

For the other inequality, note that $t \in I$ and $x \notin 2I$ implies $|x-t|$ is equivalent to $|x-d|$. By (1.2) and (1.4),

$$\begin{aligned}&\int_{\{|x-d|\ge 2\varepsilon\}} \left| \int k(x-t)g(x-t)b(t)\,dt - k(x-d)g * b(x) \right| w(x)\,dx \\ &\quad \le B \int_I |b(t)| \left(\int_{\{|x-d|\ge 2\varepsilon\}} |k(x-t) - k(x-d)|\,|g(x-d)| w(x)\,dx \right) dt \\ &\quad \le B \int_I |b(t)| \left(\int_{\{|x|\ge 2\varepsilon\}} |k(x-[t-d]) - k(x)|\,|g(x)| w(x+d)\,dx \right) dt \\ &\quad \le \frac{B}{|I|} \int_I w(x)\,dx \int_I |b(t)|\,dt \le B,\end{aligned}$$

thus proving (1.6). This completes the proof of the theorem.

REMARK. Let $w \in A_1$. By Theorem 1.4, if $a > 0$, the regular kernels $(1+|x|)^{-1-a/2}$ map H^1_w boundedly into L^1_w; to see this, set $g \equiv 1$ and note that $\int b = 0$. This implies convolution with ψ_a defines a bounded operator from H^1_w to L^1_w.

Let $w \in A_1$ and b be an H^1_w atom. Then

$$|I|^{-1/p} \|b\|_p \left(\int_I w(x)\,dx \right) \le \frac{|I|^{-1/p}}{w(I)} |I|^{1/p} w(I) \le 1. \tag{1.7}$$

This estimate is useful in the proofs of the following two results. These will help demonstrate that the operators T_β and $K_a * \cdot$ are bounded from H^1_w to L^1_w. (See [**7**] for details of the proofs.)

PROPOSITION 1.5. *Let $0 < a < 1$, $1 < p \leq 2$, and $w \in A_1$. Suppose $K(x) = k(x)g(x) \in L^\infty$ is a regular kernel with respect to w and $\|g * f\|_q \leq C\|f\|_p$. Suppose that for all $d \in \mathbf{R}$ and $s \geq 1$,*

$$\int_{\{2s \leq |x-d| \leq 2s^{1/(1-a)}\}} |k(x-d)|^p w(x)^p \, dx \leq \frac{C}{s} \left(\int_{\{|x-d| \leq s\}} w(x) \, dx \right)^p \tag{1.8}$$

and

$$\begin{aligned} &\int_{\{2s^{1/(1-a)} \leq |x-d|\}} |k(x-d)| \, |g(x-t) - g(x-d)| w(x) \, dx \\ &\qquad \leq \frac{C}{s} \int_{\{|x-d| \leq s\}} w(x) \, dx \quad \text{for } |t-d| \leq s, \end{aligned} \tag{1.9}$$

then,

$$\|K * f\|_{1,w} \leq B\|f\|_{H^1_w}.$$

The constant B depends on C and w.

PROOF. By Theorem 1.4, it suffices to prove for any atom b with support $I = [d - \varepsilon, d + \varepsilon]$ that

$$\int_{\{|x-d| \geq 2\varepsilon\}} |k(x-d)| \, |g * b(x)| w(x) \, dx \leq B < \infty. \tag{1.10}$$

When $|I| \leq 1$, argue as in the proof of Theorem 2 in [**7**], using the assumption that $K \in L^\infty$.

Suppose $|I| \geq 1$. The left side of (1.10) is bounded by

$$\begin{aligned} &\int_{\{2\varepsilon \leq |x-d| \leq 2\varepsilon^{1/(1-a)}\}} |k(x-d)| \, |g * b(x)| w(x) \, dx \\ &\qquad + \int_{\{2\varepsilon^{1/(1-a)} \leq |x-d|\}} |k(x-d)| \, |g * b(x)| w(x) \, dx. \end{aligned}$$

Using Hölder's inequality, (1.7), and (1.8), the first term is bounded by

$$\begin{aligned} &\left(\int_{\{2\varepsilon \leq |x-d| \leq 2\varepsilon^{1/(1-a)}\}} |k(x-d)|^p w(x)^p \, dx \right)^{1/p} \|g * b\|_q \\ &\qquad \leq C|I|^{-1/p} \int_{\{|x-d| \leq \varepsilon\}} w(x) \, dx \|b\|_p \leq B. \end{aligned}$$

Since $\int b = 0$, by Fubini's theorem, (1.7), and (1.9), the second term is controlled by

$$\int \left(\int_{\{2\varepsilon^{1/(1-a)} \leq |x-d|\}} |k(x-d)| \, |g(x-t) - g(x-d)| w(x) \, dx \right) |b(t)| \, dt \leq B.$$

This completes the proof of (1.10).

The analog of Proposition 1.5 for $a > 1$ is contained in

PROPOSITION 1.6. *Let $a > 1$, $1 < p \leq 2$, and $w \in A_1$. Suppose $K(x) = k(x)g(x) \in L^\infty$ is a regular kernel with respect to w and $\|g * f\|_q \leq C\|f\|_p$. Set $T(s) = \max(s, s^{1/(1-a)})$. Suppose that for all $d \in \mathbf{R}$,*

$$(1.11)\quad \int_{\{|x-d|\geq 2T(s)\}} |k(x-d)|^p w(x)^p \, dx \leq \frac{C}{s}\left(\int_{\{|x-d|\leq s\}} w(x)\, dx\right)^p, \qquad s > 0,$$

and

$$(1.12)\quad \begin{aligned} &\int_{\{2s\leq |x-d|\leq 2T(s)\}} |k(x-d)|\,|g(x-t) - g(x-d)| w(x)\, dx \\ &\qquad \leq \frac{C}{s}\int_{\{|x-d|\leq s\}} w(x)\, dx \quad \text{for } |t-d| \leq s \leq 1, \end{aligned}$$

then

$$\|K * f\|_{1,w} \leq B\|f\|_{H^1_w}.$$

The constant B depends on C and w.

PROOF. As in the previous proof, it suffices to prove (1.10). In case $|I| \geq 1$, using Hölder's inequality, (1.7), and (1.11) yields

$$\begin{aligned} &\int_{\{|x-d|\geq 2\varepsilon\}} |k(x-d)|\,|g * b(x)| w(x)\, dx \\ &\qquad \leq \left(\int_{\{|x-d|\geq 2\varepsilon\}} |k(x-d)|^p w(x)^p \, dx\right)^{1/p} \|g * b\|_q \leq B. \end{aligned}$$

Let $|I| \leq 1$. Then

$$\begin{aligned} &\int_{\{2\varepsilon\leq |x-d|\}} |k(x-d)|\,|g * b(x)| w(x)\, dx \\ &\qquad = \int_{\{2\varepsilon\leq|x-d|\leq 2\varepsilon^{1/(1-a)}\}} |k(x-d)|\,|g * b(x)| w(x)\, dx \\ &\qquad\quad + \int_{\{2\varepsilon^{1/(1-a)}<|x-d|\}} |k(x-d)|\,|g * b(x)| w(x)\, dx. \end{aligned}$$

Using (1.12) and arguing as in the proof of the previous proposition, the first term is bounded by a constant. The second term is handled like the case $|I| \geq 1$ above. This completes the proof.

In the next proposition, we consider the operators T_β and $K_a * \cdot$ acting on functions in H^1_w.

PROPOSITION 1.7. *Let $0 < a \neq 1$, $\beta = a/(a-1)$, and $w \in A_1$. There is a constant C, independent of f, such that*

$$\text{(i)} \qquad \|K_a * f\|_{1,w} \leq C\|f\|_{H^1_w}$$

and

$$\text{(ii)} \qquad \|T_\beta f\|_{1,w} \leq C\|f\|_{H^1_w}.$$

PROOF. By (1.1) and the remarks following Definition 1.3, it suffices to prove (i). Since $w \in A_1$ there is a p, $1 < p < 2$, so that $w^p \in A_1$. Decompose $K_a(x) = k_a(x)g_a(x)$, with $k_a(x) = (1+|x|)^{[(2-a)/q]-1}$ and $g_a(x) = (1+|x|)^{-(2-a)/q}e^{i|x|^a}$. Using standard arguments (see, e.g. [**7**]) and the remarks following Definition 1.3, these kernels satisfy Propositions 1.5 and 1.6. Since $\|g_a * f\|_q \leq C\|f\|_p$ (see [**7**]), inequality (i) is proved. This completes the proof of the proposition.

The next two results, proved by Stromberg and Torchinsky [**14**, **15**, **16**], will be useful for the proof of our next lemma.

THEOREM 1.8. *Let $w \in A_1$. If m is a Hörmander multiplier, then the operator T defined by $(Tf)\widehat{\ }(\xi) = m(\xi)\hat{f}(\xi)$ maps H^1_w into L^1_w.*

For example, if m satisfies the condition

$$|x|\left|\frac{dm}{dx}\right| + |m(x)| \leq C,$$

then m is a Hörmander multiplier. In fact, since these multipliers commute with the Hilbert transform, they map H^1_w into H^1_w.

THEOREM 1.9. *Let $\{T_z\}$ be an analytic family of linear operators such that T_{z_1} maps H^1_w into L^1_w and T_{z_2} maps L^2 into L^2. Let $0 < \theta < 1$. For $z(\theta) = \theta z_1 + (1-\theta)z_2$ and $1/p = \theta/1 + (1-\theta)/2$, $T_{z(\theta)}$ maps $L^p_{w^{2-p}}$ into $L^p_{w^{2-p}}$.*

Many of the standard Calderón-Zygmund-Hörmander kernels and their corresponding multipliers can be shown to satisfy Theorem 1.8 by using Theorem 1.4.

For the operators $T_{\beta,c}$ defined in the introduction we prove

LEMMA 1.10. *Let $w \in A_1$, $1 < p \leq 2$, and suppose $\beta < 0$ or $\beta > 1$. Set $c = (p-2)\beta/2p$. Then, there is a C independent of f such that*

$$\|T_{\beta,c}f\|_{p,w^{2-p}} \leq C\|f\|_{p,w^{2-p}}.$$

PROOF. Consider the analytic family of operators $\{\mathcal{T}_z\}$ defined by

$$\mathcal{T}_z f(x) = \int e^{-i\xi x}|\xi|^{-z\beta/2}e^{i|\xi|^\beta}\theta(\xi)\hat{f}(\xi)\,d\xi.$$

Note that $\mathcal{T}_1 f(x) = T_\beta f(x)$. Define g by $\hat{g}(\xi) = |\xi|^{-iy\beta/2}\hat{f}(\xi)$. Then $\mathcal{T}_{1+iy}f(x) = \mathcal{T}_1 g(x) = T_\beta g(x)$, so by Proposition 1.7 and the comment following Theorem 1.8,

$$\|\mathcal{T}_{1+iy}f\|_{1,w} = \|T_\beta g\|_{1,w} \leq B\|g\|_{H^1_w} \leq B\|f\|_{H^1_w}.$$

Since the multiplier defining $\mathcal{T}_{iy}$ is bounded,

$$\|\mathcal{T}_{iy}f\|_2 \leq B\|f\|_2.$$

The result follows from Theorem 1.9 with $y = 0$ and $\theta = (2-p)/p$.

We now prove Lemma 1.2. We use the notation f^* for the Hardy-Littlewood maximal function of a locally integrable function f.

PROOF. Using the characterization of A_1 weights in [**3**], if $w \in A_1$ then $w(x) = (g^*(x))^\delta h(x)$, where g^* is finite almost everywhere, $0 < \delta < 1$, and $h(x)$ is equivalent to a constant. Define η by $(2-p)\eta = \delta$ and set $u(x) = (g^*(x))^\eta (h(x))^{1/(2-p)}$. Then $u \in A_1$. Applying Lemma 1.10 to u completes the proof.

We will use a decomposition result for A_1 weights which is proved in [**1**]. We state the result in a form that is applicable for our purposes.

LEMMA 1.11. *Suppose $\Omega \subset \mathbf{R}$ is an open set such that $\mathbf{R} - \Omega \neq \emptyset$. Let $\{I_j\}$ be a Whitney decomposition of Ω and set $I_j^* = 9I_j/8$, $j = 1, 2, \ldots$. Then, for every $w \in A_1$ there is an A_1 weight $\rho > 0$ and a C so that*

$$C^{-1}w(x) \leq \rho(x) \leq Cw(x) \quad \textit{for all } x \in \mathbf{R} - \Omega \tag{1.13}$$

and

$$\operatorname*{ess\,sup}_{I_j^*} \rho(x) \leq C \operatorname*{ess\,inf}_{I_j} w \quad \textit{for } j = 1, 2, \ldots. \tag{1.14}$$

We conclude this section with two results about smooth cutoff functions.

PROPOSITION 1.12. *Let $\beta < 0$ or $\beta > 1$, $0 < d < 1$, and $1 < p < \infty$. Let $\psi = \psi_d$ be a C^∞ function satisfying $\psi(x) \equiv 1$ for $|x| \leq d$ and $\psi(x) \equiv 0$ for $|x| \geq 1$. Then, there is a radially decreasing function $\varphi \in L^p \cap L^1$ which is essentially constant on annuli so that*

$$\left|\left(\frac{1-\psi(\xi)}{|\xi|^{\beta/q}}\right)^{\wedge}(x)\right| \leq C\varphi(x) \quad \textit{for } \beta > 1, \tag{i}$$

$$\left|\left(\frac{\psi(\xi)}{|\xi|^{\beta/q}}\right)^{\wedge}(x)\right| \leq C\varphi(x) \quad \textit{for } \beta < 0. \tag{ii}$$

PROOF. Consider $\beta > 1$. Suppose $|x| \leq 1$. If $\beta > q$ then

$$\left(\frac{1-\psi(\xi)}{|\xi|^{\beta/q}}\right)^{\wedge}(x) = \int \frac{(1-\psi(\xi))}{|\xi|^{\beta/q}} e^{-i\xi x}\, d\xi$$

is bounded by a constant depending on d, β, and p. If $1 < \beta < q$, we decompose the integral above into

$$\int_{\{d \leq |\xi| \leq 1\}} \frac{(1-\psi(\xi))}{|\xi|^{\beta/q}} e^{-i\xi x}\, d\xi + \int_{\{|\xi| \geq 1\}} \frac{e^{-i\xi x}}{|\xi|^{\beta/q}}\, d\xi.$$

By the change of variable $u = \xi x$, it follows that the second integral is $O(1/|x|^{1-(\beta/q)})$. Using integration by parts on the first integral yields

$$\int_{\{d \leq |\xi| \leq 1\}} \frac{e^{-i\xi x}}{|\xi|^{\beta/q}}\, d\xi + \int_{\{d \leq |\xi| \leq 1\}} \psi'(\xi) \left(\int_{\{d \leq |t| \leq |\xi|\}} \frac{e^{-itx}}{|t|^{\beta/q}}\, dt\right) d\xi$$
$$= O\left(\frac{1}{|x|^{1-(\beta/q)}}\right).$$

When $\beta = q$, the integral is bounded by $\ln(1/|x|)$.

For $|x| \geq 1$, the estimate

$$\left|\frac{d^k}{d\xi^k}\left(\frac{1-\psi(\xi)}{|\xi|^\delta}\right)\right| \leq c(1+|\xi|)^{-(1+\delta)} \quad \text{for } k \geq 2$$

implies

$$\left|\left(\frac{1-\psi(\xi)}{|\xi|^\delta}\right)^{\wedge}(x)\right| \leq c|x|^{-2}.$$

Suppose $\beta < 0$. For $|x| \leq 1$, the integral defining $(\psi(\xi)/|\xi|^{\beta/q})^{\wedge}(x)$ is bounded. If $|x| > 1$, we use integration by parts to show that

$$\int_{\{|\xi| \leq 1\}} \frac{\psi(\xi)}{|\xi|^{\beta/q}} e^{-i\xi x}\, d\xi = \frac{1}{ix} \int_{\{|\xi| \leq 1\}} \left(\frac{\psi'(\xi)}{|\xi|^{\beta/q}} - \frac{\beta}{q} \frac{\psi(\xi)}{|\xi|^{(\beta/q)+1}}\right) e^{-i\xi x}\, d\xi.$$

Making the change of variables $t = \xi x$ and using further applications of integration by parts (depending on $1 + \beta/q$) complete the proof.

We can now prove

PROPOSITION 1.13. *Let $\beta < 0$ or $\beta > 1$, $d = 2^{-1/|\beta-1|}$, and $\psi(x) = \psi_d(x)$. Set $\psi_k(\xi) = \psi(\xi/2^{k/(\beta-1)})$ and define S_k, $k = 1, 2, \ldots$, by*

$$\hat{S}_k(\xi) = \begin{cases} \theta(\xi)\psi_k(\xi)e^{i|\xi|^\beta}|\xi|^{-\beta/2} & \text{for } \beta > 1, \\ \theta(\xi)(1-\psi_k(\xi))e^{i|\xi|^\beta}|\xi|^{-\beta/2} & \text{for } \beta < 0. \end{cases}$$

Then, there are constants $\alpha > 1$ and $C > 0$ so that $|S_k(x)| \le C|x|^{-\alpha}$ for all $|x| > |\beta|2^{k+2}$.

PROOF. We consider the case $\beta > 1$ since similar arguments prove the result for $\beta < 0$. An integration by parts shows

$$\begin{aligned} S_k(x) &= \int \frac{\theta(\xi)\psi_k(\xi)}{|\xi|^{\beta/2}} e^{i(|\xi|^\beta - \xi x)}\, d\xi \\ &= \int \frac{\theta(\xi)\psi_k(\xi)}{|\xi|^{\beta/2}} \frac{1}{i(\beta|\xi|^{\beta-1} - x)} \left[i(\beta|\xi|^{\beta-1} - x)e^{i(|\xi|^\beta - \xi x)} \right] d\xi \\ &= -\int \frac{(\theta'(\xi)\psi_k(\xi) + \theta(\xi)\psi_k'(\xi))}{|\xi|^{\beta/2} i(\beta|\xi|^{\beta-1} - x)} e^{i(|\xi|^\beta - \xi x)}\, d\xi \\ &\quad + \frac{\beta}{2} \int \frac{\theta(\xi)\psi_k(\xi)e^{i(|\xi|^\beta - \xi x)}}{|\xi|^{\beta/2+1} i(\beta|\xi|^{\beta-1} - x)}\, d\xi \\ &\quad + \beta(\beta-1) \int \frac{\theta(\xi)\psi_k(\xi)|\xi|^{\beta-2}}{|\xi|^{\beta/2} i(\beta|\xi|^{\beta-1} - x)^2} e^{i(|\xi|^\beta - \xi x)}\, d\xi. \end{aligned}$$

We shall estimate the first integral and omit the other cases which are similar.

Since $\theta'(\xi) = 0$ for $|\xi| \ge 1$, the first integral of the sum equals

$$\begin{aligned} &\int_{\{1/2 \le |\xi| \le 1\}} \frac{\theta'(\xi)(\beta|\xi|^{\beta-1} - x)}{|\xi|^{\beta/2} i(\beta|\xi|^{\beta-1} - x)^2} e^{i(|\xi|^\beta - \xi x)}\, d\xi \\ &\quad + \int_{E(k,\beta)} \frac{\psi_k'(\xi)(\beta|\xi|^{\beta-1} - x)}{|\xi|^{\beta/2} i(\beta|\xi|^{\beta-1} - x)^2} e^{i(|\xi|^\beta - \xi x)}\, d\xi = A + B, \end{aligned}$$

where $E(k,\beta) = \{2^{(k-1)/(\beta-1)} \le |\xi| \le 2^{k/(\beta-1)}\}$. Another integration by parts shows $|A|$ is bounded by

$$\int_{\{1/2 \le |\xi| \le 1\}} |\theta''(\xi)| \left| \int_{\{1/2 \le |t| \le |\xi|\}} \frac{\beta|t|^{\beta-1} - x}{|t|^{\beta/2} i(\beta|t|^{\beta-1} - x)^2} e^{i(|t|^\beta - tx)}\, dt \right| d\xi.$$

We apply the second mean-value theorem for integrals to the inner integral and use the fact that $|x| > |\beta|2^{k+2}$ to conclude that $|A| \le C/x^2$. Similarly, $|B|$ is bounded by

$$\begin{aligned} &|\psi_k'(\pm 2^{k/(\beta-1)})| \left| \int_{E(k,\beta)} \frac{(\beta|t|^{\beta-1} - x)e^{i(|t|^\beta - tx)}}{|t|^{\beta/2} i(\beta|t|^{\beta-1} - x)^2}\, dt \right| \\ &\quad + \int_{E(k,\beta)} |\psi_k''(\xi)| \left| \int_{\{2^{(k-1)/(\beta-1)} \le |t| \le |\xi|\}} \frac{(\beta|t|^{\beta-1} - x)e^{i(|t|^\beta - tx)}}{|t|^{\beta/2} i(\beta|t|^{\beta-1} - x)^2}\, dt \right| d\xi. \end{aligned}$$

Applying the second mean-value theorem for integrals to the inner integral and using $\psi_k'(\xi) = 2^{-k/(\beta-1)}\psi'(\xi 2^{-k/(\beta-1)})$ and $\psi_k''(\xi) = 2^{-2k/(\beta-1)}\psi''(\xi 2^{-k'(\beta-1)})$, we have

$$\begin{aligned} |B| &\le \frac{C}{2^{k\beta/2(\beta-1)}}\frac{x^{-2}}{2^{k/(\beta-1)}} + \frac{Cx^{-2}}{2^{k\beta/2(\beta-1)}}\frac{2^{k/(\beta-1)}}{2^{2k/(\beta-1)}} \\ &= \frac{2Cx^{-2}}{2^{(2+\beta)k/2(\beta-1)}} \le Cx^{-2}. \end{aligned}$$

This gives the estimate of a constant times x^{-2} as we wished to show. Similar arguments applied to $S_k(x)$ when $\beta < 0$ lead to the estimate

$$|S_k(x)| \le C/|x|^\alpha \quad \text{for } |x| \ge |\beta| 2^{k+2} \quad \text{with } \alpha = \frac{1+5|\beta|/2}{1+|\beta|} > 1.$$

2. Proof of the main theorem. We now prove Theorem 1.

PROOF. Fix $\lambda > 0$ and set $\Omega = \{x \in \mathbf{R}\colon f^*(x) > \lambda\}$. By the Whitney lemma, $\Omega = \bigcup_j I_j$, where the I_j are intervals with disjoint interiors and

$$\frac{1}{|I_j|}\int_{I_j} |f(x)|\,dx \le C\lambda.$$

Let x_j be the center of I_j and $I_j^* = 9I_j/8$ the interval concentric with I_j and $9/8$ times as long. Set $\tilde\Omega = \bigcup_j (4I_j)$ for $\beta < 0$ or $\tilde\Omega = \bigcup_j (4\beta I_j)$ for $\beta > 1$.

Let $f_j(x) = f(x)\chi_{I_j}(x)$ and write

$$f(x) = g(x) + \sum_{|I_j|\le 2} f_j(x) + \sum_{|I_j|>2} f_j(x) = g(x) + S(x) + L(x).$$

It follows that $|g(x)| \le C\lambda$ and $\|g\|_{1,w} \le C\|f\|_{1,w}$. Since $A_1 \subset A_2$, by Theorem 2.3 of [**2**] and Lemma 1.1,

$$\begin{aligned} w(\{x\colon |T_\beta g(x)| > \lambda\}) &\le \lambda^{-2}\int |T_\beta g(x)|^2 w(x)\,dx \\ &\le C\lambda^{-2}\int |g(x)|^2 w(x)\,dx \le \frac{C}{\lambda}\int |g(x)|w(x)\,dx \\ &\le \frac{C}{\lambda}\int |f(x)|w(x)\,dx. \end{aligned} \tag{2.1}$$

By the definition of Ω and the fact that A_1 weights satisfy the doubling condition

$$w(\tilde\Omega) \le C\sum_j w(I_j) = Cw(\Omega) \le \frac{C}{\lambda}\|f\|_{1,w}. \tag{2.2}$$

Thus, we need only estimate S and L on the complement of $\tilde\Omega$.

Let $J = \{j \in \mathbf{Z}^+\colon |I_j| \le 2\}$ and consider $S(x) = \sum_{j\in J} f_j(x)$. Choose $p \ne \beta'$ satisfying Lemma 1.2, $\rho(x)$ as in Lemma 1.11, and define η_β by $\hat\eta_\beta(\xi) = |\xi|^{-\beta/q}\theta(\xi)$. By Proposition 1.12, $|\eta_\beta(x)| \le \varphi(x)$, where φ is radially decreasing, essentially constant on annuli, and in $L^p \cap L^1$. Note first that

$$\begin{aligned} &w\left(\left\{x \notin \tilde\Omega\colon |T_\beta S(x)| > \lambda\right\}\right) \le \frac{C}{\lambda^p}\int_{\tilde\Omega^c} |T_\beta S(x)|^p w(x)\,dx \\ &\quad\le \frac{C}{\lambda^p}\int_{\tilde\Omega^c} |T_\beta S(x)|^p \rho(x)\,dx \le \frac{C}{\lambda^p}\int_{\mathbf{R}} |\eta_\beta * S(x)|^p \rho(x)\,dx \end{aligned}$$

by applying Lemma 1.2 to $T_\beta S = T_{\beta,c}(\eta_\beta * S)$. Writing

$$\eta_\beta * S(x) = \sum_{j\in J} \chi_{I_j^*}(x)(\eta_\beta * f_j)(x) + \sum_{j\in J} \chi_{(I_j^*)^c}(x)(\eta_\beta * f_j)(x),$$

we have

$$\begin{aligned} \frac{C}{\lambda^p}\int |\eta_\beta * S(x)|^p \rho(x)\,dx &\le \frac{C}{\lambda^p}\left(\int \left|\sum_{j\in J} \chi_{I_j^*}(x)(\eta_\beta * f_j)(x)\right|^p \rho(x)\,dx\right. \\ &\qquad \left. + \int \left|\sum_{j\in J} \chi_{(I_j^*)^c}(x)(\eta_\beta * f_j)(x)\right|^p \rho(x)\,dx\right) \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Since the I_j^*'s have bounded overlaps, we have

$$\begin{aligned} \mathrm{I} &\le \frac{C}{\lambda^p}\sum_{j\in J}\left(\sup_{x\in I_j^*}\rho(x)\right)\int_{I_j^*} |\eta_\beta * f(x)|^p\,dx \\ &\le \frac{C}{\lambda^p}\sum_{j\in J}\left(\sup_{x\in I_j^*}\rho(x)\right)\left(\int |f_j(x)|\,dx\right)^p \int_{\mathbf{R}} |\eta_\beta(y)|^p\,dy \\ &\le \frac{C}{\lambda}\sum_{j\in J}\left(\sup_{x\in I_j^*}\rho(x)\right)|I_j|^{p-1}\int |f_j(x)|\,dx \int_{\mathbf{R}} |\eta_\beta(y)|^p\,dy. \end{aligned}$$

But $|I_j| \le 2$ and $\eta_\beta \in L^p$, thus the expression above is at most

$$\frac{C}{\lambda}\sum_{j\in J}\int |f_j(x)|w(x)\,dx \le \frac{C}{\lambda}\|f\|_{1,w}.$$

To estimate II, let $x \in (I_j^*)^c$. For $y \in I_j$, $\varphi(x-y)$ is equivalent to $\varphi(x-x_j)$. Thus, with x_j the center of I_j, we have,

$$\begin{aligned} \left|\sum_{j\in J}\int \eta_\beta(x-y)f_j(y)\,dy\right| &\le \sum_{j\in J}\sup_{y\in I_j}|\eta_\beta(x-y)|\int |f_j(y)|\,dy \\ &\le C\sum_{j\in J}\sup_{y\in I_j}\varphi(x-y)\int |f_j(y)|\,dy \\ &\le C\lambda\sum_{j\in J}|I_j|\varphi(x-x_j) \\ &\le C\lambda\sum_{j\in J}\int_{I_j}\varphi(x-y)\,dy \le C\lambda\|\varphi\|_1 \le C\lambda. \end{aligned}$$

This implies

$$\begin{aligned}
\mathrm{II} &= \frac{C}{\lambda^p}\int \left|\sum_{j\in J}\chi_{(I_j^*)^c}(x)(\eta_\beta * f_j)(x)\right|^p \rho(x)\,dx \\
&\le \frac{C}{\lambda}\int \left|\sum_{j\in J}\chi_{(I_j^*)^c}(x)(\eta_\beta * f_j)(x)\right| \rho(x)\,dx \\
&\le \frac{C}{\lambda}\sum_{j\in J}\left(\int |f_j(y)|\,dy\right)\int \varphi(x-x_j)\chi_{(I_j^*)^c}(x)\rho(x)\,dx.
\end{aligned}$$

Since $\varphi \in L^1$ and radially decreasing and $\rho \in A_1$,

$$\int \varphi(x-x_j)\chi_{(I_j^*)^c}(x)\rho(x)\,dx \le C \operatorname*{ess\,sup}_{x\in I_j}\rho(x).$$

Therefore,

$$\begin{aligned}
\mathrm{II} &\le \frac{C}{\lambda}\sum_{j\in J}\left(\operatorname*{ess\,sup}_{x\in I_j}\rho(x)\right)\int |f_j(y)|\,dy \\
&\le \frac{C}{\lambda}\sum_{j\in J}\int |f_j(y)|w(y)\,dy \le \frac{C}{\lambda}\|f\|_{1,w}.
\end{aligned}$$

Combining the estimates for I and II,

$$w(\{x\in\tilde{\Omega}\colon |T_\beta S(x)|>\lambda\}) \le \frac{C}{\lambda}\|f\|_{1,w}. \tag{2.3}$$

Consider $L(x) = \sum_{|I_k|\ge 2} f_j(x)$. Set $d = 2^{-1/|\beta-1|}$ and choose ψ satisfying the conditions of Proposition 1.12. Define ψ_k by $\psi_k(\xi) = \psi(\xi/2^{k/(\beta-1)})$. Let

$$\mathcal{F}_k = \{j\colon 2^{k-1} < |I_j| \le 2^k\}, \qquad k = 2,3,\ldots,$$

and $L_k(x) = \sum_{j\in\mathcal{F}_k} f_j(x)$ so that $L(x) = \sum_{k=2}^\infty L_k(x)$.

Define R_k by

$$\hat{R}_k(\xi) = \begin{cases} 1-\psi_k(\xi), & \beta > 1, \\ \psi_k(\xi), & \beta < 0. \end{cases}$$

Recalling the definition of $\hat{S}_k$ from Proposition 1.13, we then have

$$\begin{aligned}
(T_\beta L_k)\hat{}\,(\xi) &= \theta(\xi)e^{i|\xi|^\beta}|\xi|^{-\beta/2}\hat{L}_k(\xi)\{\psi_k(\xi) + (1-\psi_k(\xi))\} \\
&= (S_k * L_k)\hat{}\,(\xi) + \{T_\beta(R_k * L_k)\}\hat{}\,(\xi).
\end{aligned}$$

Using Proposition 1.13, changing the order of integration, and arguing as in the proof of II when handling $S(x)$,

$$\begin{aligned} w&\left(\left\{x \notin \tilde{\Omega}\colon \left|\sum_{k=2}^{\infty}(S_k * L_k)(x)\right| > \lambda\right\}\right) \\ &\leq \frac{C}{\lambda}\sum_{k=2}^{\infty}\sum_{j\in\mathcal{F}_k}\int_{(\tilde{\Omega})^c}\left(\int |S_k(x-y)|\,|f_j(y)|\,dy\right) w(x)\,dx \\ &\leq \frac{C}{\lambda}\sum_{k=2}^{\infty}\sum_{j\in\mathcal{F}_k}\int_{(\tilde{\Omega})^c}\left(\int \frac{1}{|x-y|^{\alpha}}|f_j(y)|\,dy\right) w(x)\,dx \\ &\leq \frac{C}{\lambda}\sum_{k=2}^{\infty}\sum_{j\in\mathcal{F}_k}\int |f_j(y)|\left(\int_{\{|x-y|>|I_j|\}}\frac{w(x)}{|x-y|^{\alpha}}\,dx\right) dy. \end{aligned}$$

Since $|I_j| \geq 1$, the above term is at most

$$\frac{C}{\lambda}\sum_j\left(\operatorname*{ess\,inf}_{x\in I_j} w(x)\right)\int |f_j(y)|\,dy \leq \frac{C}{\lambda}\sum_j\int |f_j(y)|w(y)\,dy.$$

Hence,

$$w\left(\left\{x \notin \tilde{\Omega}\colon \left|\sum_{k=2}^{\infty}(S_k * L_k)(x)\right| > \lambda\right\}\right) \leq \frac{C}{\lambda}\|f\|_{1,w}. \tag{2.4}$$

The last term is handled similarly to $S(x)$. Defining p and η_β as before

$$\begin{aligned} w&\left(\left\{x \notin \tilde{\Omega}\colon \left|\sum_{k=2}^{\infty}T_\beta(R_k * L_k)(x)\right| > \lambda\right\}\right) \\ &\leq \frac{C}{\lambda^p}\int_{(\tilde{\Omega})^c}\left|\sum_{k=2}^{\infty}T_\beta(R_k * L_k)(x)\right|^p w(x)\,dx \\ &\leq \frac{C}{\lambda^p}\int_{(\tilde{\Omega})^c}\left|\sum_{k=2}^{\infty}T_\beta(R_k * L_k)(x)\right|^p \rho(x)\,dx \\ &\leq \frac{C}{\lambda^p}\int\left|\sum_{k=2}^{\infty}(\eta_\beta * R_k * L_k)(x)\right|^p \rho(x)\,dx \end{aligned}$$

by Lemma 1.2 applied to $T_\beta(R_k * L_k) = T_{\beta,c}(\eta_\beta * R_k * L_k)$. Write

$$\begin{aligned} (\eta_\beta * R_k * L_k)(x) = &\sum_{j\in\mathcal{F}_k}(\eta_\beta * R_k * f_j)(x)\chi_{I_j^*}(x) \\ &+ \sum_{j\in\mathcal{F}_k}(\eta_\beta * R_k * f_j)(x)\chi_{(I_j^*)^c}(x). \end{aligned}$$

Since $(\eta_\beta * R_k)\hat{}(\xi) = (1-\psi_k(\xi))|\xi|^{-\beta/q}$, making the change of variables $\xi = 2^{k/(\beta-1)}t$ in the integral for the inverse Fourier transform of $(\eta_\beta * R_k)\hat{}$ and using Proposition 1.12, we get

$$|(\eta_\beta * R_k)(x)| \leq C\frac{2^{k/(\beta-1)}}{2^{k\beta/(\beta-1)q}}\varphi(2^{k/(\beta-1)}x). \tag{2.5}$$

Since $\varphi \in L^p$, this implies

$$\int |(\eta_\beta * R_k)(x)|^p \, dx \le C 2^{-k(p-1)}.$$

The first double sum (over $2 \le k < \infty$ and $j \in \mathcal{F}_k$) is handled exactly as I for the function $S(x)$, replacing η_β by $\eta_\beta * R_k$ and using the above estimate and the fact that $|I_j| \le 2^k$ for $j \in \mathcal{F}_k$. Thus, we get

$$\begin{aligned} &\frac{C}{\lambda^p} \int \left| \sum_{k=2}^{\infty} \sum_{j \in \mathcal{F}_k} (\eta_\beta * R_k * f_j)(x) \chi_{I_j^*}(x) \right|^p \rho(x) \, dx \\ &\le \frac{C}{\lambda} \sum_{k=2}^{\infty} \sum_{j \in \mathcal{F}_k} \sup_{x \in I_j^*} \rho(x) |I_j|^{p-1} \int |f_j(x)| \, dx \int |\eta_\beta * R_k)(x)|^p \, dx \\ &\le \frac{C}{\lambda} \|f\|_{1,w}. \end{aligned} \tag{2.6}$$

The second sum is handled similarly to II, again with η_β replaced by $\eta_\beta * R_k$. In fact, to apply the argument used to estimate II, all we need is that for $k \ge 2$, $\|\eta_\beta * R_k\|_1 \le C$ and $|\eta_\beta * R_k|$ has a radially decreasing majorant. Since both of these are guaranteed by (2.5),

$$\frac{C}{\lambda^p} \int \left| \sum_{k=2}^{\infty} \sum_{j \in \mathcal{F}_k} (\eta_\beta * R_k * f_j)(x) \chi_{(I_j^*)^c}(x) \right|^p \rho(x) \, dx \le \frac{C}{\lambda} \|f\|_{1,w}$$

which combined with (2.6) yields

$$w\left(\left\{ x \notin \tilde{\Omega} \colon \left| \sum_{k=2}^{\infty} T_\beta (R_k * L_k)(x) \right| > \lambda \right\} \right) \le \frac{C}{\lambda} \|f\|_{1,w}. \tag{2.7}$$

The proof of the theorem now follows from (2.1), (2.2), (2.3), (2.4), and (2.7).

3. Weighted L^p estimates. In this section we shall discuss the kernels

$$K_{a,b+iy}(x) = (1+|x|)^{-b+iy} e^{i|x|^a}, \qquad 0 < a \ne 1, b \le 1,$$

for $x, y \in \mathbf{R}$. These kernels have been studied in [**2**, **7**].

In Theorem 3.2 of [**2**], a weighted inequality for $p = 2$ is proved in which the weights are essentially constant on annuli. Suppose $a > 1$ and $1 < p \le 2$. For weights that are essentially constant on annuli we show this condition is necessary as well as sufficient. For $p = 1$, we give a slight improvement of Lemma 4.2 of [**2**]. This can be viewed as a weighted H^1 estimate.

PROPOSITION 3.1. *Let $a > 1$ and*

$$(-\infty, \infty) = \bigcup_{n=-\infty}^{\infty} I_n,$$

where $I_n = [c_n, c_{n+1}]$ and $c_{n+1} - c_n \ge 1$ for all n. Suppose w is linear on each I_n and there is a constant C so that for all n and $x, t \in I_n$

$$0 < w(x) \le C w(t). \tag{3.1}$$

Let $\gamma \in \mathbf{R}$ and set $(U_{i\gamma}f)(x) = \int K_{a,1+iy}(x-t)f(t)(w(t))^{i\gamma}\,dt$. Then there is a constant C', independent of γ, y, and f, so that

$$\|U_{i\gamma}f\|_1 \leq C'(1+|\gamma|)(1+|y|)\|f\|_{H^1}.$$

PROOF. Let $b(t)$ be an atom supported in the interval $I = [\alpha, \alpha+\delta]$. Set $g(t) = e^{i|t|^a}(1+|t|)^{(a-2)/2}$ and $k(t) = (1+|t|)^{-(a/2)-iy}$. Following the proof of Lemma 4.2 of [**2**], we estimate

$$\psi = \int_{\{2\delta \leq |x-\alpha| \leq 2\Delta\}} |k(x-\alpha)| \left| \int_I g(x-t)b(t)(w(t))^{i\gamma}\,dt \right| dx,$$

where $0 < \delta < 1$ and $\Delta = \delta^{-1/(a-1)}$. Since $|I| = \delta < 1$, there exists an n so that $I \subset [c_{n-1}, c_n] \cup [c_n, c_{n+1}]$. In the inner integral, add and subtract $(w(r_n))^{i\gamma}$ (where $r_n = c_n$ in case both $I \cap I_{n-1} \neq \varnothing$ and $I \cap I_n \neq \varnothing$; otherwise take $r_n = \alpha$) and use the fact that $\int b = 0$ to get

$$\psi = \int_{\{2\delta \leq |x-\alpha| \leq 2\Delta\}} |k(x-\alpha)| \left| \int_I g(x-t)b(t)((w(t))^{i\gamma} - (w(r_n))^{i\gamma})\,dt \right.$$
$$\left. + (w(r_n))^{i\gamma} \int [g(x-t) - g(x-\alpha)]b(t)\,dt \right| dx.$$

By the remark following Lemma 4.2 of [**2**] it suffices to show

$$|(w(t))^{i\gamma} - (w(r_n))^{i\gamma}| \leq C|\gamma|\,|t-r_n|. \tag{3.2}$$

Note that

$$(w(t))^{i\gamma} - (w(r_n))^{i\gamma} = i\gamma \int_{w(r_n)}^{w(t)} s^{-1+i\gamma}\,ds$$

which implies the estimate

$$|(w(t))^{i\gamma} - (w(r_n))^{i\gamma}| \leq \frac{|\gamma|\,|w(t)-w(r_n)|}{\operatorname{ess\,inf}_I w}. \tag{3.3}$$

Since w is linear on I_{n-1}, $w(t) = M_{n-1}t + A$ where

$$M_{n-1} = \frac{w(c_n) - w(c_{n-1})}{c_n - c_{n-1}}.$$

By (3.1) and the fact that $c_n - c_{n-1} \geq 1$,

$$|M_{n-1}| \leq C \operatorname*{ess\,inf}_{I_{n-1}\cup I_n} w \leq C \operatorname*{ess\,inf}_I w.$$

The same argument shows that $|M_n|$ also satisfies this bound. Thus, (3.2) follows from (3.3). This completes the proof of the proposition.

We use this result to prove Theorem 4.

PROOF. We first prove the necessity of the norm inequality. Since w is essentially constant on annuli, w is defined by its values $w(2^m)$, $m = 0, 1, 2, \ldots$. Suppose the result is false so that $K_{a,b}$ is a bounded operator from L^p_w to L^p_w and

$$\limsup_{m\to\infty} \frac{w(2^m)}{2^{m(a(p-1)+p(b-1))}} = \infty.$$

This implies that for all $n = 1, 2, \ldots$ there is an $R_n > 1$ so that

$$w(x) > n(R_n)^{a(p-1)+p(b-1)} \quad \text{for } R_n \leq |x| \leq (1+2d)R_n,$$

with d chosen to satisfy Lemma 6.2 of [**2**]. Since $K_{a,b}$ is bounded on L^p_w, that lemma implies

$$(R_n)^{(1-a)p}\int_{\{R_n\le|x|\le(1+2d)R_n\}} w(x)(1+|x|)^{-bp}\,dx \le C\int_{\{|x|\le(R_n)^{1-a}\}} w(x)\,dx$$

so that

$$n(R_n)^{(1-a)p}(R_n)^{a(p-1)+p(b-1)}R_n(R_n)^{-bp}\le C(R_n)^{1-a}.$$

Therefore, $n\le C$ which contradicts the assumption that the lim sup equals ∞.

For the sufficiency, consider first the case $b=1$. The estimate on w is $1\le w(x)\le C(1+|x|)^{a(p-1)}$ for all x. Set $v(0)=w(0)$, $v(\pm 2^m)=w(\pm 2^m)$ for $m=0,1,2,\ldots$, and define $v(x)$ to be linear on the intervals $[0,1]$, $[-1,0]$, $[-2^{m+1},-2^m]$, $[2^m,2^{m+1}]$, $m=0,1,2,\ldots$.

Consider the family of operators

$$U_\tau f(x)=(v(x))^{\tau/2(p-1)}\int K_{a,1}(x-t)f(t)(v(t))^{-\tau/2(p-1)}\,dt$$

with $\tau=\alpha+i\gamma$, $\alpha,\gamma\in\mathbf{R}$, $0\le\alpha\le 1$. By definition, $v(x)>0$ satisfies the properties of Proposition 3.1; thus, we get that

$$\|U_{i\gamma}f\|_1\le C\|f\|_{H^1}.$$

In addition, since $1\le(v(x))^{1/(p-1)}\le C(1+|x|)^a$, and v is essentially constant on annuli, from Theorem 3.3 of [**2**] we have

$$\|U_{1+i\gamma}f\|_2^2=\int\left|\int K_{a,1}(x-t)f(t)(v(t))^{-1/2(p-1)}\cdot(v(t))^{-i\gamma/2(p-1)}\,dt\right|^2(v(x))^{1/(p-1)}\,dx\le C\|f\|_2^2.$$

Since $\{U_\tau\}$ is an analytic family of operators for $\tau=\alpha+i\gamma$, $0\le\alpha\le1$ and $\gamma\in\mathbf{R}$,

$$\|U_\alpha f\|_p\le C\|f\|_p$$

for $1/p=(1-\alpha)/1+\alpha/2$. This implies $\alpha/2=(p-1)/p$ so that

$$\int|K_{a,1}*f(x)|^p v(x)\,dx\le C\int|f(x)|^p v(x)\,dx.$$

By the definition of v and the conditions on w, v is equivalent to w for all x. This proves the result for $b=1$.

When $1-a/2\le b<1$, set

$$v(x)=(w(x))^{(a+2b-2)/(a(p-1)+p(b-1))}.$$

Since $1\le v(x)\le C(1+|x|)^{a+2b-2}$ and v is essentially constant on annuli, it follows from Theorem 3.3 of [**2**] that

$$\|K_{a,b}*f\|_{2,v}\le C\|f\|_{2,v}.$$

Theorem 5 of [**7**] implies for $p_0=a/(a+b-1)$,

$$\|K_{a,b}*f\|_{p_0}\le C\|f\|_{p_0}.$$

Interpolating between these two results, for $1/p = \theta/p_0 + (1-\theta)/2$ we have

$$\int |(v(x))^{(1-\theta)/2} K_{a,b} * f(x)|^p \, dx \le C \int |(v(x))^{(1-\theta)/2} f(x)|^p \, dx,$$

where

$$\frac{p(1-\theta)}{2} = \frac{a - p(a+b-1)}{a - 2(a+b-1)} = \frac{a(1-p) + p(1-b)}{2-a-2b}.$$

Therefore, $(v(x))^{p(1-\theta)/2} = w(x)$. This completes the proof of the theorem.

REMARK. For $1 - a/2 \le b < 1$ and $p = a/(a+b-1)$, we get that w is equivalent to a constant. The sufficiency for these cases follows from [**7**] while the necessity follows from Lemma 6.2 of [**2**] and the remarks following that proof. In particular, this includes the case $b = 1 - a/2$ and $p = 2$.

We would like to point out that Proposition 3.1 cannot be extended to the case $0 < a < 1$, since the operator $U_{i\gamma}$ is bounded from H^1_w into L^1_w if and only if w is a constant function.

It was shown in [**2**] that $K_{a,1}$ maps L^p_w into L^p_w for $1 < p < \infty$ and $w \in A_p$. When $b < 1$, this is no longer the case. Lemma 6.3 of [**2**] shows that $K_{a,b}$ does not define a bounded operator on $L^p_{(1+|x|)^\alpha}$ for $\alpha > p - 1 + p(b-1)/a$ and hence there are A_p weights for which $K_{a,b}$ does not define a bounded operator. The following theorem shows that the result of Lemma 6.3 of [**2**] is sharp.

THEOREM 3.2. *Let $0 < a < 1$, $1 - a/2 \le b < 1$, and $a/(a+b-1) \le p \le a/(1-b)$. Then*

$$\|K_{a,b} * f\|_{p,(1+|x|)^\alpha} \le C \|f\|_{p,(1+|x|)^\alpha}$$

if and only if

$$-1 + \frac{q}{a}(1-b) \le \alpha \le p - 1 + \frac{p}{a}(b-1).$$

PROOF. The necessity of the range follows from the remarks preceding the theorem and duality. To show the sufficiency, it is enough to consider the case $\alpha = p - 1 + p(b-1)/a$; the other values follow from duality and interpolation with change of measures. Further, we need only prove the result for $p_0 = a/(a+b-1)$ and $p_1 = a/(1-b)$. The intermediate values of p then follow by interpolation with change of measures. The norm inequality for p_0 is a consequence of the facts that $K_{a,b}$ maps L^p into L^p for $a/(a+b-1) \le p \le a/(1-b)$ and $\alpha = p_0 - 1 + p_0(b-1)/a = 0$. To see the result for p_1, note that Lemma 4.5 of [**2**] is true for $0 < a < 1$ and when $p_1 = a/(1-b)$, $bp_1 + a - 2 = p_1 - 2 = p_1 - 1 + p_1(b-1)/a$.

REMARK. We note that this result is not true for $b = 1$. A consequence of Lemma 6.3 of [**2**] is that the norm inequality fails if $\alpha = p - 1$.

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